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**Stringy Model for QCD at Finite Density
and
Generalized Hagedorn Temperature ^a**

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Abstract

Using generic properties of string theories, we show how interesting non-perturbative features of QCD can be exploited in heavy ion collisions. In particular, a generalized “semi-circle” law for the phase diagram in the temperature-chemical potential plane is derived.

1 Introduction

It is the belief of many that experimental studies of ultra-relativistic nuclear collisions in the near future should reach sufficiently high energy densities so as to reveal the physics of deconfined quarks and gluons. More recently, the behavior of matter at high baryon density has also received much attention. There are growing theoretical indications that there is a rich phase structure for QCD in the temperature-chemical potential, $T - \mu$, plane. For instance, for $SU(2)$ flavor, one believes that the associated chiral-symmetry restoration transition is second order for μ small, and it turns into first order as μ is increased. On the other hand, for $SU(3)$ flavor, the transition is always the first order. The

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“conjectured” confinement-deconfinement phase diagrams^{1,2} for these two cases are shown in Fig. 1(a) and Fig. 1(b).

Instead of dealing with features which depend on the specific flavor degrees of freedom, we focus on “generic features” which can be elucidated by exploiting the “stringy” aspect of QCD spectrum in the confined regime. In particular, we would like to suggest, based on lessons from string studies^{3,4,5,6,7} that interesting non-perturbative features of QCD can be learned at deconfinement energy densities for non-zero values of chemical potential. In particular, for heavy ion collisions, we explain how a stringy representation allows us to derive a generic expression for the deconfinement curve in the $T - \mu$ plane, which has a similar structure as exhibited in Fig. 1(a) and Fig. 1(b).

There are two key ingredients involved: **Confining force leading to an exponentially growing spectrum, and global symmetries emerging through compactification of extra spatial dimensions.** We first review how the notion of Hagedorn temperature arises in a stringy theory for $\mu = 0$. We next show how additive quantum numbers naturally arise in string theory through toroidal compactification. The question of phase diagram in the $T - \mu$ plane will be discussed last.

2 Strings at High Energy Densities

Let us begin by first considering the situation where $\mu = 0$. Assuming thermo-equilibrium can be achieved in hadronic multi-particle production, one expects that the energy density \mathcal{E} can be parameterized monotonically in terms of a temperature, T . At low temperature, $\mathcal{E}(T)$ can be given effectively in terms of pion gas. At high temperature and after deconfinement, one has Stefan-Boltzmann law, (appropriate for gluons and light quarks), which, for $SU(2)$ flavor, leads to $\mathcal{E} \simeq 12T^4$. Indeed, this expectation has been substantiated by various lattice calculations.⁹ These “numerical experiments” suggest that the deconfinement density is of the order $\mathcal{E}_d \simeq 12T_0^4$, where $T_0 \sim 150 - 200\text{MeV}$. We would like to stress that for energy densities near and/or below \mathcal{E}_d , the effective degrees of freedom for QCD are “string-like”. Because of the confining force, string excitations can be characterized by rising linear Regge trajectories, leading to an exponentially growing particle spec-

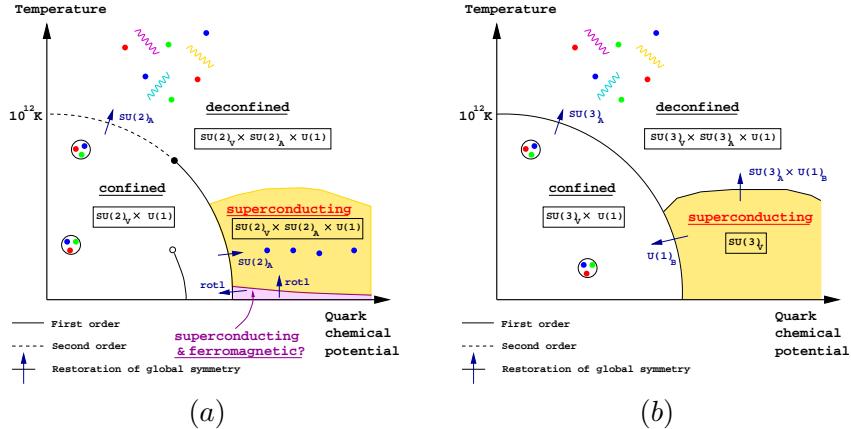


Figure 1: Conjectured phase diagram for QCD: (a) with two massless flavors, (b) with three massless flavors.

trum. Approaching from the confined phase, deconfinement is signaled by a “Hagedorn temperature”, $T_H \simeq T_0$. Probing this kinematical regime should reveal interesting non-perturbative features of confined QCD at hadronic scales. [Atick-Witten have suggested⁵ that $T_0 < T_H$, and we have previously considered⁶ the possibility that $T_H < T_0$. In what follows, we shall assume $T_H \sim T_0$.]

2.1 Statistical Mechanics of Strings

For statistical systems with a finite number of fundamental degrees of freedom, it is well-known that a microcanonical and a canonical descriptions are equivalent. For strings, this equivalence breaks down at high energy densities. However, it turns out that the canonical partition function remains useful when considered as an analytic function of the inverse temperature, $\beta \equiv 1/T$.

The fundamental quantity in a canonical approach is the partition

function:

$$Z(\beta, V) \equiv \text{Tr} e^{-\beta \hat{H}} = \sum_{\alpha} e^{-\beta E_{\alpha}}, \quad (1)$$

where the sum is over all possible multiparticle states of the system. For a microcanonical approach, one works with a density function, which counts the number of micro-states, $\Omega(E, V) dE \equiv \sum_{\alpha} \delta(E - E_{\alpha}) dE$. Statistical mechanics based on a microcanonical ensemble is more general, even though it is often more convenient to work with canonical ensemble, *e.g.*, when interactions must be included.

Representing the Dirac- δ function by an integral along imaginary axis, we find that $\Omega(E, V) = \sum_{\alpha} \int_{-i\infty}^{+i\infty} \frac{d\beta}{2\pi i} e^{\beta(E-E_{\alpha})}$. If one can deform the contour into a region where interchanging the order of sum and integral is allowed, one obtains

$$\Omega(E, V) = \int_{\beta_0-i\infty}^{\beta_0+i\infty} \frac{d\beta}{2\pi i} Z(\beta, V) e^{\beta E}. \quad (2)$$

The allowed region is labeled by the interception of the contour with the real axis, β_0 . One can then recover canonical partition function from the microcanonical function via $Z(\beta, V) = \int_0^{\infty} dE \Omega(E, V) e^{-\beta E}$, which provides an alternative analytic definition for $Z(\beta, V)$. In order to determine the allowed contour, β_0 , we need to examine **the analytic property in β** for the canonical partition function, $Z(\beta, V)$.

2.2 Single-Particle Density and Hagedorn Temperature

Formulating a consistent *effective string theory* for QCD has been one of the major challenges for string theorists.¹⁰ Since we are still far from accomplishing this task, any insight into the problem, either theoretical or experimental, can be useful. A common feature of all string-like theories is the rapid increase of mass degeneracies. We shall assume that the desired effective QCD string theory has an asymptotic exponential mass degeneracy, which characterizes the growth of its effective degrees of freedom. This can be exhibited by calculating the single-particle density at high energies $f(E, V) = V \sum_i d(m_i) \int d^D p \delta \left(E - \sqrt{p^2 + m_i^2} \right)$,

where $d(m_i)$ is the degeneracy at mass-level m_i . More directly, we can work with the inverse Laplace transform

$$\tilde{f}(\beta, V) = V \sum_i d(m_i) \int d^D p e^{-\beta \sqrt{p^2 + m_i^2}}, \quad (3)$$

The contour at β_0 will be determined by the right-most singularity in β for $\tilde{f}(\beta, V)$. This will be explained in greater detail in the next section, and, in particular, on how this generalizes in the presence of non-zero chemical potential.

To be more specific, for an ideal gas of strings under the Maxwell-Boltzmann (MB) statistics, one has $Z(\beta, V) \simeq e^{\tilde{f}(\beta, V)} - 1$. It can be shown that $\tilde{f}(\beta, V)$ is analytic for $\text{Re } \beta$ sufficiently large and its right-most singularity is at $\beta = \beta_H > 0$, *i.e.*,

$$\tilde{f}(\beta, V) \sim V [h(\beta)(\beta - \beta_H)^\gamma + \lambda(\beta)], \quad (4)$$

where $\gamma > 0$ and both $h(\beta)$ and $\lambda(\beta)$ are analytic around and to the right of β_H . Given $Z(\beta, V)$ as an analytic function of β , the microcanonical function $\Omega(E, V)$ can be recovered through Eq. (2), with $\beta_0 > \beta_H$. That is, *the totality of physics of microcanonical approach for free strings has been encoded in the analyticity of $Z(\beta, V)$.*^{3,4} In particular, one finds that

$$f(E, V) \sim V E^{-(\gamma+1)} e^{\beta_H E}. \quad (5)$$

That is, we can characterize a string theory at high energy density minimally by two exponents, γ and β_H , (the latter is often referred to as the inverse Hagedorn temperature.)

We have considered elsewhere, in a quantum statistical treatment, analytic property of $Z(\beta, V)$ in β for various string theories.^{3,4} We find that, generically, $Z(\beta, V)$ is analytic for $\text{Re } \beta > 0$ except at isolated points. For each string theory, because of the exponential growth in mass degeneracy, there is always an isolated rightmost singularity at $\beta = \beta_H$, *i.e.*, the inverse Hagedorn temperature for that theory. There is a finite gap in their real parts between β_H and the next singularity to the left, and this gap is theory-dependent but calculable.

For conventional systems, Eq. (2) can often be approximated by a saddle point contribution at β^* . For strings, at low energies, β^* lies

to the right of β_H . Under such a condition, the temperature is related to E by $E = -\frac{\partial \log Z}{\partial \beta}|_{\beta^*}$. However, as the energy density $\mathcal{E} \equiv E/V$ is raised, one reaches a point where either the saddle point moves to the left of β_H , or it gets close to β_H that the fluctuations about the saddle point become large. When this occurs, the saddle point approximation to Eq. (2) breaks down, and it defines the density scale \mathcal{E}_H for the region of interest to heavy ion collisions. Correspondingly, $\beta_H^{-1} = T_H$ will be identified with the deconfining temperature T_0 .⁶

It is worth noting that, for $\mathcal{E} > \mathcal{E}_H$, whereas it is no longer meaningful to speak of a Boltzmann temperature, the statistical mechanics of free strings is still given unambiguously by Eq. (2). One can in fact push the contour in (2) to the left of the singular point, β_H , by a finite distance η , $\eta > 0$. As one moves past this point, one picks up an additional contribution involving the discontinuity across the cut. Denoting the discontinuity by $\Delta Z(\beta, V)$, $\beta < \beta_H$, the large- E behavior of $\Omega(E, V)$ is dominated by the singularity at β_H

$$\Omega(E, V) = - \int_{\beta_H - \eta}^{\beta_H} \frac{d\beta}{2\pi i} \Delta Z(\beta, V) e^{\beta E} + O(e^{(\beta_H - \eta)E}), \quad \eta > 0. \quad (6)$$

Once $\Delta Z(\beta, V)$ is known, the dominant behavior of $\Omega(E, V)$ can be found. Therefore, the large- E limit of a free-string theory can best be approached by working first with the canonical quantity, $Z(\beta, V)$. Applications to relevant physical systems have been discussed elsewhere^{3,4}.

3 Counting Effective Degrees of Freedom at Hadronic Scales

The basic degrees of freedom of a string theory are string excitations. Each excitation can be given a particle attribute, *i.e.*, a mass m . The mass is due to internal string oscillations; as such, it can take on increasing values, with a corresponding increase in degeneracies. Let us illustrate this with the simplest possible bosonic string model.

A classical string is an extended object which “lives” in space and can be characterized by a set of coordinate functions, $\{X^i(\tau, \sigma)\}$, where $\sigma \in [0, \pi]$ and $i = 1, 2, \dots$ specify the set of spatial directions. Let us concentrate on “closed strings”, *i.e.*, $X(\tau, \sigma + \pi) = X(\tau, \sigma)$. It follows that the motion can be represented by a Fourier expansion, $X^i = x_0^i +$

$\sum_{n=1}^{\infty} (x_n^i \cos 2n\sigma + \bar{x}_n^i \sin 2n\sigma)$, where $\{x_n^i\}$ describes the motion of the CM of the string. The set of $\{x_n^i\}$ and $\{\bar{x}_n^i\}$ describe amplitudes for the internal vibration of the string. Motion for this “free” string can be described by an action:

$$S = \int \frac{d\tau}{4\alpha'} \sum_i \left\{ (\dot{x}_0^i)^2 + \sum_{n=1} \left[((\dot{x}_n^i)^2 - n^2(x_n^i)^2) + (x_n \leftrightarrow \bar{x}_n) \right] \right\}. \quad (7)$$

For the internal motions, they therefore can also be described by an infinite set of harmonic oscillators, with oscillation frequencies increasing as $n = 1, 2, 3, \dots$.

One can next turn to quantization. However, there is the problem of relativistic invariance which must be taken care of. It is of course tempting to maintain covariance all the way. To this end, we need to introduce x^μ , where $\mu = 0, 1, 2, \dots, D$, where D is the spatial dimension, (*e.g.*, for the real world, $D = 3$.) This should be done for both the CM and the internal motions. However, one finds that not all these variables are independent, and one must impose proper gauge constraints in quantization. Or equivalently, one can choose $x^0 \equiv \tau$ as the time and work in light-cone gauge where the independent variables are simply the “transverse” coordinates: $x_0^i, x_n^i, \bar{x}_n^i$, where $i = 1, 2, 3, \dots, D - 1$.

Classically, one can have separate “left-moving” and “right-moving” vibrations, *i.e.*, independent wave forms of the types $x(\tau + \sigma)$ and $x(\tau - \sigma)$. It is more convenient to treat these modes separately. For each right-moving (or left-moving) oscillator mode, we can introduce creation and annihilation operators, \tilde{a}_n^i and $\tilde{a}_n^{i\dagger}$, or $(a_n^i$ and $a_n^{i\dagger})$. The Fock space can then be specified by eigenvalues, $0, 1, 2, \dots$, for the corresponding number operators, \tilde{N}_n^i and N_n^i .

If we denote the D th direction as “longitudinal”, the dispersion relation for each stationary state can be written as that for a particle $E^2 = p_L^2 + \vec{p}^2 + m^2$, $m^2 = m_l^2 + m_r^2$, where

$$m_l^2 = \frac{1}{\alpha'} \left[-\delta_l + \sum_{i=1}^{D-1} \sum_{n=1} \left(n \tilde{N}_n^i \right) \right], \quad m_r^2 = \frac{1}{\alpha'} \left[-\delta_r + \sum_{i=1}^{D-1} \sum_{n=1} \left(n N_n^i \right) \right]. \quad (8)$$

For bosonic string theory, $\delta_l = \delta_r = 1$. That is, when one identifies each string mode as a particle, the mass originates from the internal string

vibrations. One should also add that translation invariance in σ leads to another constraint, $m_l^2 = m_r^2$. The details for the mass spectrum will of course vary from theory to theory. However, generic features for all string models include: (i) equal-spacing rule in m^2 for the mass spectrum, and (ii) exponentially increasing mass degeneracies. To be specific, writing $m^2 = \alpha' N$, one finds that the degeneracy factor for a closed string is given as a product of left- and right-moving factors, $d(m) = d_l(m_l)d_r(m_r)$, *i.e.*, $d_l(m_l) = \sum_{\tilde{N}_n^i}^{\infty} \delta(N/2 + \delta_l - \sum_i \sum_n n (\tilde{N}_n^i))$, and similarly for $d_r(m_r)$. They both increase with \sqrt{N} exponentially.

With these ingredients, one finds that¹¹

$$\tilde{f}(\beta, V) \sim V\beta \int_E d^2\tau \tau_2^{-(D+3)/2} D_l(\bar{z}) D_r(z) e^{-\beta^2/4\pi\alpha'\tau_2}, \quad (9)$$

where the integration in $\tau = \tau_1 + i\tau_2$ is over a half-strip region $E : -1/2 < \tau_1 < 1/2, \tau_2 > 0$. With $\bar{z} = \exp(-2\pi i\bar{\tau})$ and $z = \exp(2\pi i\tau)$, D_l and D_r are generating functions for left- and right-mass degeneracies. By studying the convergence of this integral at $\tau_2 = 0$ as β is lowered, the nature of the Hagedorn singularity at β_H can be identified.^{3,11}

4 Hagedorn Temperature for $\mu \neq 0$

We need to first address the question of “additive” quantum numbers, which indicate the presence of $U(1)$ global symmetries. This comes most naturally in a string theory by a toroidal compactification, which gives rise to “winding numbers”, w_i , and “discrete momenta”, m_i , one set for each compactified direction. These discrete quantum numbers are conserved in any process where strings are joined or split. They form the generic representations for conserved charges in a string theory.

If we compactify the j th transverse direction on a circle of radius R , it follows that, for the CM motion, the requirement $e^{ip^j x_0^j} = e^{ip^j(x_0^j + 2\pi R)}$ leads to a discrete momentum, $p^j = m_j/R$, m_j an arbitrary integer. Similarly, the periodicity condition now generalizes to $x^j(\sigma + \pi) = x^j(\sigma) + 2\pi w_j R$, where w_j is an integral winding number. That is, for each compactified direction, $x^j(\tau, \sigma)$, we can quantize the theory over a

“solitonic” background,

$$x^j(\tau, \sigma) = \left(\frac{m_j}{R} \right) \tau + (2w_j R) \sigma. \quad (10)$$

This provides an extra contribution to the mass term: $(1/2)(m_j/R + w_j R/\alpha')^2$ for m_l^2 and $(1/2)(m_j/R - w_j R/\alpha')^2$ for m_r^2 respectively. The constraint, $m_l^2 = m_r^2$, remains.

4.1 Statistical Mechanics at Finite Energy and Charge Densities

In a canonical approach, the partition function becomes

$$Z(\beta, \mu, V) \equiv \text{Tr} e^{-\beta(\hat{H} + \mu \hat{Q})} \equiv \text{Tr} e^{-\beta \hat{H} - \bar{\mu} \hat{Q}} = \sum_{\alpha} e^{-\beta E_{\alpha} - \bar{\mu} Q_{\alpha}}, \quad (11)$$

where the sum is over all possible multiparticle states of the system. Note that we have re-scaled the chemical potential so that $\bar{\mu}$ is dimensionless. In a microcanonical approach, for a system with fixed E and Q , one has $\Omega(E, Q, V) dE \equiv \sum_{\alpha} \delta_{Q, Q_{\alpha}} \delta(E - E_{\alpha}) dE$. Following a similar analysis as that for $\mu = 0$, we find that

$$\Omega(E, Q, V) = \int_{-i\pi}^{i\pi} \frac{d\bar{\mu}}{2\pi i} \int_{\beta_0(\bar{\mu}) - i\infty}^{\beta_0(\bar{\mu}) + i\infty} \frac{d\beta}{2\pi i} Z(\beta, \mu, V) e^{\beta E + \bar{\mu} Q}. \quad (12)$$

The allowed region is again labeled by the interception of the contour with the real axis, $\beta_0(\bar{\mu})$.

The analyticity of $Z(\beta, \mu, V)$ can again be determined by the single-particle function, which can be symbolically written as $\tilde{f}(\beta, \bar{\mu}, V) = \sum_a e^{-\beta \epsilon_a - \bar{\mu} q_a}$. Let us be more precise. For each conserved charge, a chemical potential should be introduced. For our toroidally compactified strings, $\bar{\mu} q$ actually represents $\bar{\nu}_i m_i + \bar{\mu}_i \omega_i$, where m_i and ω_i are momenta and winding numbers.

4.2 Generalized Hagedorn Singularity and Phase Diagram

We have computed

$$\tilde{f}(\beta, \bar{\mu}, \bar{\nu}, V) \sim V \beta \int_E d^2 \tau \tau_2^{-(D'+3)/2} D_l(\bar{z}) D_r(z) e^{(-\beta^2/4\pi\alpha'\tau_2)} W(\bar{z}, z), \quad (13)$$

where the additional factor due to compactification is

$$W(\bar{z}, z) = \prod'_i \sum_{m_i, w_i} \bar{z}^{\frac{1}{2}(\frac{m_i}{R} + w_i \bar{R})^2} z^{\frac{1}{2}(\frac{m_i}{R} - w_i \bar{R})^2} e^{-(\bar{\nu}_i m_i + \bar{\mu}_i w_i)}, \quad (14)$$

where $\bar{R} \equiv \alpha'^{-1/2} R$, and the product is over the compactified directions. Again, by examining the divergence at $\tau_2 \rightarrow 0$ as one lowers β , one finds that the generalized Hagedorn temperature is given by

$$\frac{2\beta_H(\mu, \nu)}{(\alpha')^{\frac{1}{2}}} = \left[\bar{\omega}_l + \sum_i \left(\bar{\nu}_i \bar{R} + \frac{\bar{\mu}_i}{\bar{R}} \right)^2 \right]^{1/2} + \left[\bar{\omega}_r + \sum_i \left(\bar{\nu}_i \bar{R} - \frac{\bar{\mu}_i}{\bar{R}} \right)^2 \right]^{1/2}. \quad (15)$$

In particular, $\beta_H(0, 0) = (2\pi^2 \alpha')^{\frac{1}{2}} (\sqrt{\bar{\omega}_l} + \sqrt{\bar{\omega}_r}) = \beta_H$, ($\bar{\omega}_{l,r} = 8\pi^2 \omega_{l,r}$), is simply the inverse Hagedorn temperature identified previously. For standard string theories, $\omega_{l,r}$ take on values 1 or 2.

We are now in the position to apply our result to the case of QCD at finite density. Here, we are interested in the phase diagram for the deconfinement transition in the $T - \mu$ plane. We therefore consider the simplest possible situation where we set all μ_i and ν_i to be zero, except one, which will be referred to as the “baryonic chemical potential”.

In order for us to make use of the above result, Eq. (15), we must recognize that $\bar{\mu} = \beta\mu$. (Alternatively, we could have worked directly with μ without involving the reduced chemical potential $\bar{\mu}$. We have explicitly verified that the same result would be reached. In particular, in the large volume limit, the $\bar{\mu}$ -integration in Eq. (12) is dominated by a saddle point with μ real.¹²⁾ Let us denote the phase boundary by $\beta_H(\mu) \equiv T(\mu)^{-1}$. It follows that

$$\frac{2}{\sqrt{\alpha'}} = \left[\bar{\omega}_l T(\mu)^2 + \left(\frac{\mu}{\bar{R}} \right)^2 \right]^{1/2} + \left[\bar{\omega}_r T(\mu)^2 + \left(\frac{\mu}{\bar{R}} \right)^2 \right]^{1/2}, \quad (16)$$

which corresponds to a “distorted” semi-circle law. To be explicit, consider the case $\bar{\omega}_l = \bar{\omega}_r$. One finds that $\left(\frac{T(\mu)}{T_H} \right)^2 + \left(\frac{\mu}{\mu_0} \right)^2 = 1$. Note that $\mu_0 = \alpha'^{-1} R$, which is precisely the mass for one unit of baryonic charge. This result can be expressed more simply as

$$T(\mu) = T_H \sqrt{1 - \frac{\mu^2}{\mu_0^2}}, \quad (17)$$

as depicted in Fig. 1(a) and (b).

5 Remarks

It is well understood that the character of QCD changes depending on the nature of available probes. At short distances, the basic degrees of freedom are quarks and gluons. As one moves to larger distance scales, the QCD coupling increases and one enters the non-perturbative regime. Short of resulting to lattice Monte Carlo studies, the most promising tool for a non-perturbative treatment of QCD which builds in naturally quark-gluon confinement remains the large- N expansion. In this approach, although the vacuum of QCD at hadronic scales is complicated, model studies suggest that the effective degrees of freedom of QCD can most profitably be expressed in terms of “extended objects”. Indeed, low-lying hadron spectrum suggests that they can be understood as “string excitations”. In high-energy soft hadronic collisions⁸ where the interactions are mostly peripheral, it is possible to “see” the dominant string excitations in terms of the exchanges of high-lying Regge trajectories having a closed string color topology.

Here we have concentrated on deriving the phase-boundary for the deconfining transition at non-zero baryon chemical potential. More interestingly, our approach can provide a framework for exploring interesting non-perturbative features of QCD at deconfinement energies for non-zero baryon densities through our microcanonical approach, Eq. (12). This will be discussed in a separate publication.¹²

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